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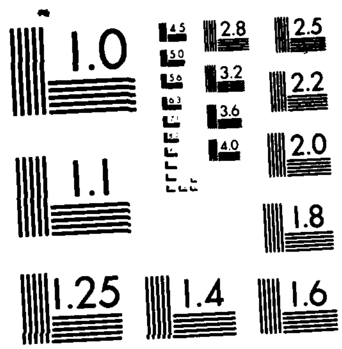

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AN EXACT TEST FOR THE NESTING EFFECT'S VARIANCE COMPONENT IN AN UNBALANCED RANDOM TWO-FOLD NESTED MODEL

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An exact test is presented for testing the significance of the nesting effect's variance components in an unbalanced random two-fold nested model. The power of this test is compared against power values obtained under other approximate tests.

An Exact Test for the Nesting Effect's Variance Component in an Unbalanced Random Two-Fold Nested Model

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Abstract. This paper presents an exact test concerning the nesting effect's variance component in an unbalanced random two-fold nested classification model. The test requires that the total number of observations exceeds 2b-1, where b is the total number of levels of the nested factor.

Keywords. Variance components, random effects, unbalanced nested models, hypothesis testing, power of a test.

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1. Introduction

There is no exact test for testing the significance of σ_α^2, the nesting effect's variance component in an unbalanced random two-fold nested classification model. There are, however, approximate tests which utilize ratios of mean squares (see Cummings and Caylor 1974, Tietjen 1974, and Tietjen and Moore 1968). In general, the exact distributions associated with these tests are complicated which makes it difficult to adequately determine the tests' true levels of significance and power values. A comparison of four approximate tests concerning σ_α^2 was recently made by Tan and Cheng (1984).

In this paper we present an exact F-test for testing the null hypothesis

$$H_0:\sigma_\alpha^2 = 0 \text{ versus } H_a:\sigma_\alpha^2 \neq 0. \tag{1.1}$$

Aside from the usual assumptions concerning the random effects in the model, the only other condition for the validity of the test is that $N > 2b - 1$, where N is the total number of observations and b is the total number of levels of the nested factor. The proposed test is compared against the approximate tests described in Tan and Cheng (1984) with respect to power. The results of this comparison indicate that the former test is quite efficient.

2. The development of the exact test

Consider the unbalanced two-fold nested model

$$y_{ijk} = \mu + \alpha_i + \beta_{ij} + \epsilon_{ijk}, \tag{2.1}$$

$i = 1,2,\ldots,a$; $j = 1,2,\ldots,b_i$; $k = 1,2,\ldots,n_{ij}$, where μ is an unknown constant parameter, α_i and β_{ij} are random effects associated with the nesting factor and the nested factor, respectively, and ϵ_{ijk} is a random error. We assume that α_i, β_{ij}, and ϵ_{ijk} are independently distributed as $N(0,\sigma_\alpha^2)$, $N(0,\sigma_\beta^2)$, and $N(0,\sigma_\epsilon^2)$, respectively. We also assume that

$$N > 2b - 1, \tag{2.2}$$

where $N = \sum_{i,j} n_{ij}$, $b = \sum_{i=1}^{a} b_i$. The need for inequality (2.2) will be seen later. We note that the latter assumption is quite reasonable and can, for example, be satisfied if $n_{ij} \geq 2$ for all i,j.

Let $\bar{y}_{ij} = \sum_{k=1}^{n_{ij}} y_{ijk}/n_{ij}$ $(i = 1,2,\ldots,a;\ j = 1,2,\ldots,b_i)$. From (2.1) we have

$$\bar{y}_{ij} = \mu + \alpha_i + \beta_{ij} + \bar{\varepsilon}_{ij}, \tag{2.3}$$

$i = 1,2,\ldots,a$; $j = 1,2,\ldots,b_i$, where $\bar{\varepsilon}_{ij} = \sum_{k=1}^{n_{ij}} \varepsilon_{ijk}/n_{ij}$. Model (2.3) can be rewritten in the matrix form

$$\bar{\underset{\sim}{y}} = \mu \underset{\sim}{1}_b + \underset{\sim}{A}_1\underset{\sim}{\alpha} + \underset{\sim}{I}_b\underset{\sim}{\beta} + \bar{\underset{\sim}{\varepsilon}}, \tag{2.4}$$

where $\bar{\underset{\sim}{y}}$ and $\bar{\underset{\sim}{\varepsilon}}$ are vectors consisting of the $\bar{y}_{ij}$'s and the $\bar{\varepsilon}_{ij}$'s, respectively, $\underset{\sim}{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_a)'$, $\underset{\sim}{\beta} = (\beta_{11},\beta_{12},\ldots,\beta_{ab_a})'$, $\underset{\sim}{1}_b$ is a vector of ones of dimension b, $\underset{\sim}{I}_b$ is the identity matrix of order b×b, and $\underset{\sim}{A}_1 = \mathrm{Diag}(\underset{\sim}{1}_{b_1},\underset{\sim}{1}_{b_2},\ldots,\underset{\sim}{1}_{b_a})$ is a block-diagonal matrix of vectors of ones. From (2.4) the variance-covariance matrix of $\bar{\underset{\sim}{y}}$ is

$$\mathrm{Var}\ \bar{\underset{\sim}{y}} = \hat{\underset{\sim}{A}}_1\sigma_\alpha^2 + \underset{\sim}{I}_b\sigma_\beta^2 + \underset{\sim}{K}\sigma_\varepsilon^2, \tag{2.5}$$

where

$$\hat{\underset{\sim}{A}}_1 = \underset{\sim}{A}_1\underset{\sim}{A}_1' = \bigoplus_{i=1}^{a} \underset{\sim}{J}_{b_i}, \tag{2.6}$$

$$\underset{\sim}{K} = \mathrm{Diag}(n_{11}^{-1}, n_{12}^{-1},\ldots,n_{ab_a}^{-1}), \tag{2.7}$$

where $\underset{\sim}{J}_{b_i}$ is a matrix of ones of order $b_i \times b_i$ $(i = 1,2,\ldots,a)$, and $\oplus$ denotes the direct sum. The residual sum of squares for the model in (2.1) is $T = \sum_{i,j,k}(y_{ijk} - \bar{y}_{ij})^2$. It is known that T/σ_ε^2 has the chi-squared distribution with N − b degrees of freedom independently of $\bar{\underset{\sim}{y}}$. We can write T as

$$T = \underset{\sim}{y}'\ \underset{\sim}{R}\ \underset{\sim}{y}, \tag{2.8}$$

where $\underset{\sim}{y}$ is the vector of N observations and $\underset{\sim}{R}$ is the matrix

$$\underset{\sim}{R} = \underset{\sim}{I}_N - \bigoplus_{i,j} (\underset{\sim}{J}_{n_{ij}}/n_{ij}). \tag{2.9}$$

We note that $\underset{\sim}{R}$ is symmetric and idempotent of rank $N - b$, and by assumption, $N > 2b - 1$, that is, $N - b > b - 1$. We can thus express $\underset{\sim}{R}$ as

$$\underset{\sim}{R} = \underset{\sim}{C}\, \underset{\sim}{\Lambda}\, \underset{\sim}{C}', \tag{2.10}$$

where $\underset{\sim}{C}$ is an orthogonal matrix and $\underset{\sim}{\Lambda}$ is a diagonal matrix whose first $N - b$ diagonal elements are equal to unity and the remaining b elements are equal to zero. Furthermore, we can partition $\underset{\sim}{C}$ and $\underset{\sim}{\Lambda}$ as

$$\underset{\sim}{\Lambda} = \mathrm{Diag}(\underset{\sim}{I}_{\nu_1}, \underset{\sim}{I}_{\nu_2}, \underset{\sim}{0}), \tag{2.11}$$

$$\underset{\sim}{C} = [\underset{\sim}{C}_1 : \underset{\sim}{C}_2 : \underset{\sim}{C}_3], \tag{2.12}$$

where

$$\begin{aligned} \nu_1 &= b - 1, \\ \nu_2 &= N - 2b + 1, \end{aligned} \tag{2.13}$$

and $\underset{\sim}{C}_1, \underset{\sim}{C}_2, \underset{\sim}{C}_3$ are of orders $N \times \nu_1$, $N \times \nu_2$, and $N \times b$, respectively. Note that

$$\begin{aligned} \underset{\sim}{C}_i' \underset{\sim}{C}_i &= \underset{\sim}{I}, \quad i = 1,2,3, \\ \underset{\sim}{C}_i' \underset{\sim}{C}_j &= \underset{\sim}{0}, \quad i \neq j \end{aligned} \tag{2.14}$$

Formula (2.10) can then be rewritten as

$$\underset{\sim}{R} = \underset{\sim}{C}_1 \underset{\sim}{C}_1' + \underset{\sim}{C}_2 \underset{\sim}{C}_2', \tag{2.15}$$

which results in a partitioning of the residual sum of squares T into T_1 and T_2, where

$$T_1 = \underset{\sim}{y}'\, \underset{\sim}{C}_1 \underset{\sim}{C}_1'\, \underset{\sim}{y}, \tag{2.16}$$

$$T_2 = \underset{\sim}{y}'\, \underset{\sim}{C}_2 \underset{\sim}{C}_2'\, \underset{\sim}{y}. \tag{2.17}$$

Consider now the matrix $\hat{\underset{\sim}{A}}_1$ in (2.6). There exists an orthogonal matrix $\underset{\sim}{P}$ of order $b \times b$ such that $\underset{\sim}{P}\, \hat{\underset{\sim}{A}}_1\, \underset{\sim}{P}' = \underset{\sim}{\Lambda}_1$, where

$$\underset{\sim}{\Lambda}_1 = \mathrm{Diag}(b_1, b_2, \ldots, b_a, \underset{\sim}{0}), \tag{2.18}$$

where $\underset{\sim}{0}$ is a zero matrix of order (b-a)×(b-a). This is because $\hat{\underset{\sim}{A}}_1$ has the eigenvalues $b_1, b_2, \ldots, b_a$, and 0 of multiplicity b-a. The first a rows of $\underset{\sim}{P}$ are orthonormal eigenvectors of $\hat{\underset{\sim}{A}}_1$ that correspond to $b_1, b_2, \ldots, b_a$. Let $\underset{\sim}{P}_1$ be the a×b matrix consisting of these rows, that is,

$$\underset{\sim}{P}_1 = \mathrm{Diag}(\underset{\sim}{1}'_{b_1}/\sqrt{b_1}, \underset{\sim}{1}'_{b_2}/\sqrt{b_2}, \ldots, \underset{\sim}{1}'_{b_a}/\sqrt{b_a}). \tag{2.19}$$

Let $\underset{\sim}{P}_2$ be the (b-a)×b matrix consisting of the remaining b-a rows of $\underset{\sim}{P}$. Then $\underset{\sim}{P} = [\underset{\sim}{P}'_1 : \underset{\sim}{P}'_2]'$. If $\underset{\sim}{z} = \underset{\sim}{P}\,\bar{\underset{\sim}{y}}$, then from (2.5) we have

$$\mathrm{Var}\ \underset{\sim}{z} = \underset{\sim}{\Lambda}_1 \sigma^2_\alpha + \underset{\sim}{I}_b \sigma^2_\beta + \underset{\sim}{P}\,\underset{\sim}{K}\,\underset{\sim}{P}'\sigma^2_\varepsilon. \tag{2.20}$$

Theorem 2.1. There exists an orthogonal matrix $\underset{\sim}{Q}$ of order b×b such that the first row of $\underset{\sim}{Q}\,\underset{\sim}{P}$ is $\underset{\sim}{1}'_b/\sqrt{b}$.

Proof. Define the unit vector $\underset{\sim}{e}'_1 = [\underset{\sim}{c}'_1 : \underset{\sim}{0}']$ where $\underset{\sim}{c}'_1 = (\sqrt{b_1}, \sqrt{b_2}, \ldots, \sqrt{b_a})/\sqrt{b}$ and $\underset{\sim}{0}'$ is a zero vector of dimension b-a. Then $(\underset{\sim}{I}_b - \underset{\sim}{e}_1\underset{\sim}{e}'_1)\underset{\sim}{e}_1 = \underset{\sim}{0}$. The matrix $\underset{\sim}{I}_b - \underset{\sim}{e}_1\underset{\sim}{e}'_1$ is idempotent of rank b-1. Let $\underset{\sim}{Q}_1$ denote a matrix of order b×(b-1) and rank b-1 whose columns are obtained via a Gram-Schmidt orthonormalization of the columns of $\underset{\sim}{I}_b - \underset{\sim}{e}_1\underset{\sim}{e}'_1$. Let $\underset{\sim}{Q} = [\underset{\sim}{e}_1 : \underset{\sim}{Q}_1]'$, then $\underset{\sim}{Q}$ is an orthogonal matrix and the first row of $\underset{\sim}{Q}\,\underset{\sim}{P}$, namely $\underset{\sim}{e}'_1\underset{\sim}{P}$, is $\underset{\sim}{1}'_b/\sqrt{b}$.

From Theorem 2.1 we conclude that if $\underset{\sim}{u} = \underset{\sim}{Q}'_1\underset{\sim}{z}$, where $\underset{\sim}{z} = \underset{\sim}{P}\,\bar{\underset{\sim}{y}}$ and $\underset{\sim}{Q}_1$ is the matrix described in the proof of the theorem, then

$$E(\underset{\sim}{u}) = \mu \underset{\sim}{Q}'_1\,\underset{\sim}{P}\,\underset{\sim}{1}_b = \underset{\sim}{0}, \tag{2.21}$$

since $\underset{\sim}{Q}\,\underset{\sim}{P}$ is orthogonal, and

$$\mathrm{Var}\ \underset{\sim}{u} = \underset{\sim}{Q}'_1\underset{\sim}{\Lambda}_1\underset{\sim}{Q}_1\sigma^2_\alpha + \underset{\sim}{I}_{b-1}\sigma^2_\beta + \underset{\sim}{Q}'_1\underset{\sim}{P}\,\underset{\sim}{K}\,\underset{\sim}{P}'\underset{\sim}{Q}_1\sigma^2_\varepsilon, \tag{2.22}$$

since $\underset{\sim}{Q}'_1\underset{\sim}{Q}_1 = \underset{\sim}{I}_{b-1}$. We note that $\underset{\sim}{Q}'_1\underset{\sim}{\Lambda}_1\underset{\sim}{Q}_1$ is of rank a-1. To show this we partition $\underset{\sim}{Q}'_1$ as $[\underset{\sim}{Q}'_{11} : \underset{\sim}{Q}'_{12}]$, where $\underset{\sim}{Q}'_{11}$ is (b-1)×a and $\underset{\sim}{Q}'_{12}$ is (b-1)×(b-a). Then $\underset{\sim}{Q}'_1\underset{\sim}{\Lambda}_1\underset{\sim}{Q}_1 = \underset{\sim}{Q}'_{11}\mathrm{Diag}(b_1, b_2, \ldots, b_a)\underset{\sim}{Q}_{11}$ (see 2.18). Hence, rank $(\underset{\sim}{Q}'_1\underset{\sim}{\Lambda}_1\underset{\sim}{Q}_1)$ =

rank $(\underset{\sim}{Q}_{11})$ = rank $(\underset{\sim}{Q}_{11}\underset{\sim}{Q}_{11}')$ = rank $(\underset{\sim}{I}_a - \underset{\sim}{c}_1\underset{\sim}{c}_1')$ = a-1, where $\underset{\sim}{c}_1'$ is the unit vector described in the proof of Theorem 2.1. The one before last equality follows because the columns of the matrix $[\underset{\sim}{c}_1 : \underset{\sim}{Q}_{11}]'$ are orthonormal. It follows that there exists an orthogonal matrix $\underset{\sim}{S}$ of order (b-1)×(b-1) such that

$$\underset{\sim}{Q}_1'\underset{\sim}{\Lambda}_1\underset{\sim}{Q}_1 = \underset{\sim}{S}\ \mathrm{Diag}(\underset{\sim}{D},\underset{\sim}{0})\ \underset{\sim}{S}', \tag{2.23}$$

where $\underset{\sim}{D}$ is an (a-1)×(a-1) diagonal matrix of nonzero eigenvalues of $\underset{\sim}{Q}_1'\underset{\sim}{\Lambda}_1\underset{\sim}{Q}_1$ and $\underset{\sim}{0}$ is a zero matrix of order (b-a)×(b-a).

Consider the vector $\underset{\sim}{\omega} = \underset{\sim}{S}'\underset{\sim}{u}$. From (2.21), (2.22), and (2.23) it can be seen that $\underset{\sim}{\omega}$ has a zero mean and a variance-covariance matrix given by

$$\mathrm{Var}\ \underset{\sim}{\omega} = \mathrm{Diag}(\underset{\sim}{D},\underset{\sim}{0})\sigma_\alpha^2 + \underset{\sim}{I}_{b-1}\sigma_\beta^2 + \underset{\sim}{L}\sigma_\varepsilon^2, \tag{2.24}$$

where $\underset{\sim}{L} = \underset{\sim}{S}'\underset{\sim}{Q}_1'\underset{\sim}{P}\ \underset{\sim}{K}\ \underset{\sim}{P}'\underset{\sim}{Q}_1\underset{\sim}{S}$. Define the vector $\underset{\sim}{\Omega}$ as

$$\underset{\sim}{\Omega} = \underset{\sim}{\omega} + (\lambda_{max}\underset{\sim}{I}_{b-1} - \underset{\sim}{L})^{\frac{1}{2}}\ \underset{\sim}{C}_1'\underset{\sim}{y}, \tag{2.25}$$

where λ_{max} is the largest eigenvalue of the symmetric matrix $\underset{\sim}{L}$ and $\underset{\sim}{C}_1$ is the N×(b-1) matrix in (2.12). Note that the matrix $\lambda_{max}\underset{\sim}{I}_{b-1} - \underset{\sim}{L}$ is positive semidefinite, hence $(\lambda_{max}\underset{\sim}{I}_{b-1} - \underset{\sim}{L})^{\frac{1}{2}}$ is well defined with eigenvalues equal to the square roots of the eigenvalues of $\lambda_{max}\underset{\sim}{I}_{b-1} - \underset{\sim}{L}$. Let $\underset{\sim}{\Omega}$ be partitioned as $\underset{\sim}{\Omega} = [\underset{\sim}{\Omega}_\alpha' : \underset{\sim}{\Omega}_\beta']'$, where $\underset{\sim}{\Omega}_\alpha$ and $\underset{\sim}{\Omega}_\beta$ are of dimensions a-1 and b-a, respectively.

Theorem 2.2.

(i) $E\underset{\sim}{\Omega}_\alpha = E\underset{\sim}{\Omega}_\beta = \underset{\sim}{0}$.

(ii) $\underset{\sim}{\Omega}_\alpha$ and $\underset{\sim}{\Omega}_\beta$ are statistically independent and normally distributed with the following variance-covariance matrices:

$$\mathrm{Var}\ \underset{\sim}{\Omega}_\alpha = \underset{\sim}{D}\sigma_\alpha^2 + (\sigma_\beta^2 + \lambda_{max}\sigma_\varepsilon^2)\ \underset{\sim}{I}_{a-1}, \tag{2.26}$$

$$\mathrm{Var}\ \underset{\sim}{\Omega}_\beta = (\sigma_\beta^2 + \lambda_{max}\sigma_\varepsilon^2)\ \underset{\sim}{I}_{b-a}. \tag{2.27}$$

Proof.

(i) This is true because $E(\underset{\sim}{\omega}) = \underset{\sim}{S}'E(\underset{\sim}{u}) = \underset{\sim}{0}$ by (2.21), and $E(\underset{\sim}{C}_1'\underset{\sim}{y}) = \underset{\sim}{0}$.

(ii) Since $\underset{\sim}{\omega} = \underset{\sim}{S}' \underset{\sim}{Q}_1' \underset{\sim}{P} \bar{\underset{\sim}{y}}$, then $\underset{\sim}{\Omega}$ is a linear function of the vector of observations, hence it is normally distributed. We now claim that $\underset{\sim}{\omega}$ and $\underset{\sim}{C}_1'\underset{\sim}{y}$ in (2.25) are statistically independent. To show this we write $\bar{\underset{\sim}{y}}$ in the form $\bar{\underset{\sim}{y}} = \underset{\sim}{G}\, \underset{\sim}{y}$ where $\underset{\sim}{G} = \text{Diag}(\underset{\sim}{1}'_{n_{11}}/n_{11}, \underset{\sim}{1}'_{n_{12}}/n_{12}, \ldots, \underset{\sim}{1}'_{n_{ab_a}}/n_{ab_a})$. Since $\bar{\underset{\sim}{y}}$ is statistically independent of the residual sum of squares, T, in (2.8), then $\underset{\sim}{G}\, \underset{\sim}{\Sigma}\, \underset{\sim}{R} = \underset{\sim}{0}$, where $\underset{\sim}{\Sigma} = \text{Var}\, \underset{\sim}{y}$ (see Searle 1971, p. 59). Using the representation (2.15) for $\underset{\sim}{R}$ we obtain

$$\underset{\sim}{G}\, \underset{\sim}{\Sigma}\, (\underset{\sim}{C}_1\underset{\sim}{C}_1' + \underset{\sim}{C}_2\underset{\sim}{C}_2') = \underset{\sim}{0}. \tag{2.28}$$

If we multiply (2.28) on the right by $\underset{\sim}{C}_1$ and note (2.14) we get

$$\underset{\sim}{G}\, \underset{\sim}{\Sigma}\, \underset{\sim}{C}_1 = \underset{\sim}{0}. \tag{2.29}$$

From (2.29) it follows that $\text{Cov}(\underset{\sim}{\omega}, \underset{\sim}{y}'\underset{\sim}{C}_1) = \text{Cov}(\underset{\sim}{S}'\underset{\sim}{Q}_1' \underset{\sim}{P} \bar{\underset{\sim}{y}}, \underset{\sim}{y}'\underset{\sim}{C}_1) = \underset{\sim}{S}'\underset{\sim}{Q}_1' \underset{\sim}{P}\, \text{Cov}(\bar{\underset{\sim}{y}}, \underset{\sim}{y}'\underset{\sim}{C}_1) = \underset{\sim}{S}'\underset{\sim}{Q}_1'\underset{\sim}{P}\, \text{Cov}(\underset{\sim}{G}\, \underset{\sim}{y}, \underset{\sim}{y}'\underset{\sim}{C}_1) = \underset{\sim}{S}'\underset{\sim}{Q}_1'\underset{\sim}{P}\, \underset{\sim}{G}\, \underset{\sim}{\Sigma}\, \underset{\sim}{C}_1 = \underset{\sim}{0}$. Hence, the variance-covariance matrix of $\underset{\sim}{\Omega}$ in (2.25) is of the form

$$\text{Var}\, \underset{\sim}{\Omega} = \text{Var}\, \underset{\sim}{\omega} + (\lambda_{max}\underset{\sim}{I}_{b-1} - \underset{\sim}{L})^{\frac{1}{2}}\, \underset{\sim}{C}_1'\, \underset{\sim}{\Sigma}\, \underset{\sim}{C}_1 (\lambda_{max}\underset{\sim}{I}_{b-1} - \underset{\sim}{L})^{\frac{1}{2}}\ . \tag{2.30}$$

We claim that $\underset{\sim}{C}_1'\, \underset{\sim}{\Sigma}\, \underset{\sim}{C}_1 = \sigma_\varepsilon^2\, \underset{\sim}{I}_{b-1}$. This follows from the fact that T/σ_ε^2 has the chi-squared distribution, hence $\underset{\sim}{R}\, \underset{\sim}{\Sigma}\, \underset{\sim}{R}\, \underset{\sim}{\Sigma} = \sigma_\varepsilon^2\, \underset{\sim}{R}\, \underset{\sim}{\Sigma}$ (see 2.8 and Theorem 2 in Searle 1971, p. 57), which can also be written as $\underset{\sim}{R}\, \underset{\sim}{\Sigma}\, \underset{\sim}{R} = \sigma_\varepsilon^2\, \underset{\sim}{R}$. By noting (2.15) we get

$$(\underset{\sim}{C}_1\underset{\sim}{C}_1' + \underset{\sim}{C}_2\underset{\sim}{C}_2')\, \underset{\sim}{\Sigma}\, (\underset{\sim}{C}_1\underset{\sim}{C}_1' + \underset{\sim}{C}_2\underset{\sim}{C}_2') = \sigma_\varepsilon^2\, (\underset{\sim}{C}_1\underset{\sim}{C}_1' + \underset{\sim}{C}_2\underset{\sim}{C}_2'). \tag{2.31}$$

If we now multiply (2.31) on the left by $\underset{\sim}{C}_1'$ and on the right by $\underset{\sim}{C}_1$ and note (2.14) we obtain the desired result.

From (2.24) and (2.30) we conclude that

$$\text{Var}\, \underset{\sim}{\Omega} = \text{Diag}(\underset{\sim}{D}, \underset{\sim}{0})\, \sigma_\alpha^2 + \underset{\sim}{I}_{b-1}\sigma_\beta^2 + \underset{\sim}{L}\, \sigma_\varepsilon^2 + (\lambda_{max}\underset{\sim}{I}_{b-1} - \underset{\sim}{L})\, \sigma_\varepsilon^2.$$

$$= \mathrm{Diag}(\underset{\sim}{D},\underset{\sim}{0})\ \sigma_\alpha^2 + (\sigma_\beta^2 + \lambda_{max}\ \sigma_\varepsilon^2)\ \underset{\sim}{I}_{b-1}. \tag{2.32}$$

Since Var $\underset{\sim}{\Omega}$ is a diagonal matrix, $\underset{\sim}{\Omega}_\alpha$ and $\underset{\sim}{\Omega}_\beta$ must be statistically independent. Furthermore, from (2.32) it can be concluded that these random vectors have the variance structure described in (2.26) and (2.27), respectively.

From Theorem 2.2 we can then state that

$$\underset{\sim}{\Omega}_\alpha' \ (\underset{\sim}{D}\sigma_\alpha^2 + \delta\ \underset{\sim}{I}_{a-1})^{-1}\underset{\sim}{\Omega}_\alpha \sim \chi_{a-1}^2$$

$$\underset{\sim}{\Omega}_\beta' \ \underset{\sim}{\Omega}_\beta/\delta \sim \chi_{b-a}^2,$$

where $\delta = \sigma_\beta^2 + \lambda_{max}\sigma_\varepsilon^2$. Under $H_0: \sigma_\alpha^2 = 0$, $\underset{\sim}{\Omega}_\alpha'\underset{\sim}{\Omega}_\alpha/\delta \sim \chi_{a-1}^2$, hence $F = MS_\alpha/MS_\beta$ has the central F-distribution with a-1 and b-a degrees of freedom, where $MS_\alpha = \underset{\sim}{\Omega}_\alpha'\underset{\sim}{\Omega}_\alpha/(a-1)$ and $MS_\beta = \underset{\sim}{\Omega}_\beta'\underset{\sim}{\Omega}_\beta/(b-a)$. It is easy to verify that

$$E(MS_\alpha) = \delta + \Big[\sum_{i=1}^{a-1} d_i/(a-1)\Big]\sigma_\alpha^2$$

$$E(MS_\beta) = \delta, \tag{2.33}$$

where d_i is the i^{th} diagonal element of $\underset{\sim}{D}(i=1,2,\ldots,a-1)$. Hence, large values of the test statistic F are significant.

3. A comparison of the exact test against the approximate tests in Tan and Cheng (1984)

Tan and Cheng (1984) compared the powers of four approximate tests of the hypothesis H_0 described in (1.1). The corresponding test statistics were denoted by F_1, F_2, F_3, and F_4, respectively, and their power values were derived approximately by using Laguerre polynomial expansions of the true null and non-null distributions of $F_i (i=1,2,3,4)$. The power values were obtained for different values of σ_α^2 (=.5,1,3), several combinations of the nuisance parameters σ_β^2 and σ_ε^2(=1,2,3), and for two unbalanced nested designs which we reproduce in Table 1.

In Tables 2 and 3 we give a listing of the power values associated with F_1, F_2, F_3, and F_4 as reported in Tan and Cheng (1984, Table 4, pp. 197-198) for a level of significance $\alpha = .10$.

At the α-level of significance, the power function for the exact test proposed in Section 2 is given by

$$P(MS_\alpha / MS_\beta > F_{\alpha,a-1,b-a} | H_a), \tag{3.1}$$

where H_a is the alternative hypothesis in (1.1), and $F_{\alpha,a-1,b-a}$ denotes the upper α% point of the F-distribution with a-1 and b-a degrees of freedom. Under H_a, MS_α and MS_β are independently distributed and $(b-a)MS_\beta/\delta$ is distributed as χ^2_{b-a}, but $(a-1)MS_\alpha/\delta$ no longer has the chi-squared distribution. In this case, since $\underset{\sim}{\Omega}_\alpha$ is normally distributed with a zero mean and a variance-covariance matrix Var $\underset{\sim}{\Omega}_\alpha = \underset{\sim}{D}\sigma^2_\alpha + \delta \underset{\sim}{I}_{a-1}$, $\underset{\sim}{\Omega}'_\alpha \underset{\sim}{\Omega}_\alpha = (a-1)MS_\alpha$ is distributed as $\sum_{i=1}^{a-1} \lambda_i W_i$, where the W_i's are independent chi-squared variates with one degree of freedom, and λ_i is the i^{th} eigenvalue of Var $\underset{\sim}{\Omega}_\alpha$, that is, $\lambda_i = d_i \sigma^2_\alpha + \delta$ with d_i being the i^{th} diagonal element of $\underset{\sim}{D}(i=1,2,\ldots,a-1)$ (see Johnson and Kotz 1970, p. 151). Thus, under H_a the exact test statistic F can be written as

$$F = \frac{\underset{\sim}{\Omega}'_\alpha \underset{\sim}{\Omega}_\alpha}{\delta(a-1)MS_\beta/\delta} = \frac{\sum_{i=1}^{a-1} \lambda_i W_i}{\delta(a-1)MS_\beta/\delta} . \tag{3.2}$$

Approximate values of the power function in (3.1) can be conveniently obtained by using Hirotsu's (1979, pp. 578-579) approximation of the upper probability values of a statistic of the form

$$H = \frac{\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x}/(cf)}{\hat{\sigma}^2/\sigma^2}, \tag{3.3}$$

where $\underset{\sim}{x}$ is normally distributed with a zero mean and a variance-covariance matrix $\underset{\sim}{V}$, $\underset{\sim}{A}$ is a nonnegative matrix, $\hat{\sigma}^2/\sigma^2$ is distributed as $(1/f_2)\chi^2_{f_2}$ independently of $\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x}$, and c and f are given by

$$c = \frac{1}{2}\,\kappa_2(\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x})/\kappa_1(\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x})$$

$$f = 2\,\kappa_1^2\,(\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x})/\kappa_2(\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x}), \tag{3.4}$$

where $\kappa_i(\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x})$ denotes the i^{th} cumulant of $\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x}$. For convenience we reproduce the formula for $P(H > h)$ as given in Hirotsu (1979, formula 2.4):

$$P(H > h) \approx P(F_{f,f_2} > h) + \left[\Delta/\{3(f+2)(f+4)B(\tfrac{1}{2}\,f,\ \tfrac{1}{2}\,f_2)\}\right](1+fh/f_2)^{-\frac{1}{2}(f+f_2)} \times (fh/f_2)^{\frac{1}{2}f}\left[(f+2)/(f+4) + \frac{2(f+f_2)(f+4)}{1+f_2/(fh)} + \frac{(f+f_2+2)(f+f_2)}{\{1+f_2/(fh)\}^2}\right], \tag{3.5}$$

where F_{f,f_2} denotes the F-distribution with f and f_2 degrees of freedom, $B(m_1,m_2)$ denotes the beta function, and

$$\Delta = \frac{1}{2}\left[\kappa_1(\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x})\ \kappa_3(\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x})/\kappa_2^2(\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x})\right] - 1. \tag{3.6}$$

The approximation described in (3.5) was developed via a Laguerre polynomial expansion of the true distribution of the statistic $\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x}/(2c)$ and was reported in Hirotsu (1979) to be quite satisfactory.

From (3.1) and (3.2), the power function for the exact test statistic F can be written as

$$P\left(\frac{\underset{\sim}{\Omega}_\alpha'\underset{\sim}{\Omega}_\alpha/(cf)}{MS_\beta/\delta} > \frac{\delta(a-1)}{cf}\,F_{\alpha,a-1,b-a}\,\middle|\,H_a\right), \tag{3.7}$$

where c and f are given as in (3.4), but with $\underset{\sim}{\Omega}_\alpha'\underset{\sim}{\Omega}_\alpha$ substituted for $\underset{\sim}{x}'\underset{\sim}{A}\,\underset{\sim}{x}$, that is,

$$c = \frac{\mathrm{tr}\{(\mathrm{Var}\ \underset{\sim}{\Omega}_\alpha)^2\}}{\mathrm{tr}(\mathrm{Var}\ \underset{\sim}{\Omega}_\alpha)} = \frac{\sum_{i=1}^{a-1}(d_i\sigma_\alpha^2 + \delta)^2}{\sum_{i=1}^{a-1}(d_i\sigma_\alpha^2 + \delta)} = \frac{\delta\sum_{i=1}^{a-1}(d_i\theta + 1)^2}{\sum_{i=1}^{a-1}(d_i\theta + 1)} \tag{3.8}$$

$$f = \frac{\{\mathrm{tr}(\mathrm{Var}\ \underset{\sim}{\Omega}_\alpha)\}^2}{\mathrm{tr}\{(\mathrm{Var}\ \underset{\sim}{\Omega}_\alpha)^2\}} = \frac{\{\sum_{i=1}^{a-1}(d_i\sigma_\alpha^2 + \delta)\}^2}{\sum_{i=1}^{a-1}(d_i\sigma_\alpha^2 + \delta)^2} = \frac{\{\sum_{i=1}^{a-1}(d_i\theta+1)\}^2}{\sum_{i=1}^{a-1}(d_i\theta+1)^2}, \tag{3.9}$$

where

$$\theta = \frac{\sigma_\alpha^2}{\delta} = \frac{\sigma_\alpha^2}{\sigma_\beta^2 + \lambda_{max}\sigma_\varepsilon^2} \tag{3.10}$$

(see Hirotsu 1979, p. 579). We note that the statistic which appears in (3.7) is of the same form as the statistic H in (3.3). This can be seen by taking $\underset{\sim}{x} = \underset{\sim}{\Omega}_\alpha$, $\underset{\sim}{A} = \underset{\sim}{I}_{a-1}$, and $\hat{\sigma}^2/\sigma^2 = MS_\beta/\delta$, which is distributed as $(1/f_2)\chi^2_{f_2}$ with f_2 = b-a degrees of freedom. It follows that the power value in (3.7) can be approximately computed by applying formula (3.5) and remembering that

$$H = \frac{\underset{\sim}{\Omega}_\alpha' \underset{\sim}{\Omega}_\alpha/(cf)}{MS_\beta/\delta},$$

$$h = \frac{\delta(a-1)}{cf} F_{\alpha,a-1,b-a} = \frac{(a-1)F_{\alpha,a-1,b-a}}{\sum_{i=1}^{a-1}(d_i\theta+1)} \text{ (using formulas 3.8 and 3.9),}$$

$$f_2 = b-a,$$

c and f are as given in (3.8) and (3.9), and Δ is as described in (3.6), which can also be written as

$$\Delta = \frac{[\mathrm{tr}(\mathrm{Var}\ \underset{\sim}{\Omega}_\alpha)][\mathrm{tr}\{(\mathrm{Var}\ \underset{\sim}{\Omega}_\alpha)^3\}]}{[\mathrm{tr}\{(\mathrm{Var}\ \underset{\sim}{\Omega}_\alpha)^2\}]^2} - 1$$

$$= \frac{[\sum_{i=1}^{a-1}(d_i\theta+1)][\sum_{i=1}^{a-1}(d_i\theta+1)^3]}{[\sum_{i=1}^{a-1}(d_i\theta+1)^2]^2} - 1, \tag{3.11}$$

where θ is given in (3.10) (see Hirotsu 1979, p. 579). It is interesting to note that in (3.5) the power of the exact test depends on $\sigma_\alpha^2, \sigma_\beta^2$, and σ_ε^2 through θ. A complete determination of this power requires finding the values of λ_{max}, $d_1, d_2, \ldots, d_{a-1}$, which depend on the design used, and a specification of the level of significance and the ratios $\sigma_\alpha^2/\sigma_\varepsilon^2$, $\sigma_\beta^2/\sigma_\varepsilon^2$ of the variance components.

Power values associated with the exact test statistic F were computed at the $\alpha = .10$ level of significance. The upper probability values of the F-distribution (with f and f_2 degrees of freedom) in (3.5) were obtained by using the IMSL (International Mathematical and Statistical Libraries) MDFDRE Subroutine which allows fractional degrees of freedom. The same designs and combinations of variance components as the ones used by Tan and Cheng (1984) were considered here. The results are given in Tables 2 and 3 for designs 1 and 2, respectively. From these tables it can be seen that the exact test is more efficient than the approximate tests based on the statistics F_1, F_2, F_3, and F_4. Furthermore, the exact test has the advantage that its critical value does not depend on the values of the unknown nuisance parameters σ_β^2 and σ_ε^2. This property is not shared by the approximate tests which may require good estimates of σ_β^2 and σ_ε^2 in order for their results to be reliable (see Tan and Cheng 1984, p. 194).

4. Concluding remarks

The vector $\underset{\sim}{\Omega}$ defined in (2.25) is the key to the construction of the exact test proposed in Section 2. It is a linear combination of the random vectors $\underset{\sim}{\omega}$ and $\underset{\sim}{C}_1'\underset{\sim}{y}$, where $\underset{\sim}{\omega}$ is a linear transform of the vector $\bar{\underset{\sim}{y}}$ of response means, and $\underset{\sim}{C}_1'\underset{\sim}{y}$ makes up a portion of the residual sum of squares (see formula 2.16). The power study in Section 3 clearly indicates that the exact test can be at least as efficient as the other approximate tests.

A procedure similar to the one described in this paper was effectively used to obtain exact tests concerning the main effects' variance components in an unbalanced random two-way model with interaction. Details of that procedure are given in Khuri (1985).

Table 1

Two unbalanced nested designs for the model in (2.1)

Design 1	Design 2
$a=5$	$a=4$
$b_1=2$	$b_1=1,\ b_2=2$
$b_2=b_3=b_4=b_5=1$	$b_3=3,\ b_4=4$
$n_{11}=n_{12}=4$	$n_{11}=4$
$n_{21}=n_{31}=n_{41}=n_{51}=2$	$n_{21}=n_{22}=3$
	$n_{31}=n_{32}=n_{33}=2$
	$n_{41}=n_{42}=n_{43}=n_{44}=1$

Table 2

Power values of the exact and approximate tests for Design 1 at the α = .10 level of significance.

		$\sigma_\alpha^2 = .5$				$\sigma_\alpha^2 = 1.0$				$\sigma_\alpha^2 = 3.0$			
σ_β^2	σ_ε^2	F_1	F_2	F_4	Exact	F_1	F_2	F_4	Exact	F_1	F_2	F_4	Exact
1.0	1.0	.099	.097	.098	.118	.099	.097	.094	.134	.102	.113	.101	.184
1.0	2.0	.099	.099	.097	.114	.098	.101	.095	.126	.098	.129	.097	.167
1.0	3.0	.099	.101	.096	.111	.097	.105	.094	.121	.097	.136	.096	.156
2.0	1.0	.097	.153	.099	.111	.097	.154	.099	.121	.098	.164	.10	.156
2.0	2.0	.10	.098	.099	.109	.099	.097	.098	.118	.10	.098	.097	.148
2.0	3.0	.102	.066	.098	.108	.101	.064	.097	.116	.101	.061	.095	.142
3.0	1.0	.10	.099	.099	.108	.10	.099	.099	.116	.101	.098	.10	.142
3.0	2.0	.10	.099	.099	.107	.10	.098	.099	.114	.10	.098	.098	.137
3.0	3.0	.10	.099	.099	.106	.098	.098	.098	.112	.099	.097	.096	.134

Table 3

Power values of the exact and approximate tests for Design 2 at the α = .10 level of significance

		$\sigma_\alpha^2 = .5$				$\sigma_\alpha^2 = 1.0$				$\sigma_\alpha^2 = 3.0$			
σ_β^2	σ_ε^2	F_1	F_2	F_3	Exact	F_1	F_2	F_3	Exact	F_1	F_2	F_3	Exact
1.0	1.0	.150	.159	.147	.212	.211	.229	.203	.316	.440	.473	.424	.577
1.0	2.0	.139	.141	.135	.174	.185	.194	.177	.248	.374	.427	.353	.472
1.0	3.0	.130	.131	.127	.155	.166	.173	.159	.212	.323	.347	.302	.402
2.0	1.0	.127	.131	.125	.174	.158	.166	.155	.248	.301	.323	.294	.472
2.0	2.0	.123	.126	.122	.155	.149	.158	.147	.212	.248	.296	.263	.402
2.0	3.0	.120	.119	.118	.143	.143	.145	.139	.189	.251	.267	.238	.353
3.0	1.0	.118	.118	.117	.155	.138	.140	.137	.212	.236	.245	.231	.402
3.0	2.0	.115	.117	.115	.143	.134	.138	.132	.189	.221	.236	.215	.353
3.0	3.0	.115	.117	.114	.136	.131	.136	.130	.174	.210	.227	.203	.316

References

Cummings, W. B. and D. W. Gaylor (1974), Variance Component Testing in Unbalanced Nested Designs, J. Amer. Statist. Assoc. 69, 765-771.

Hirotsu, C. (1979), An F Approximation and Its Application, Biometrika 66, 577-584.

Johnson, N.L. and S. Kotz (1970), Continuous Univariate Distributions -2 (Houghton Mifflin, Boston).

Khuri, A. I. (1985), Exact Tests for the Main Effects' Variance Components in an Unbalanced Random Two-Way Model. Technical Report No. 245, Department of Statistics, University of Florida, Gainesville, FL.

Searle, S. R. (1971), Linear Models (Wiley, New York).

Tan, W. Y. and S. S. Cheng (1984), On Testing Variance Components in Three-Stages Unbalanced Nested Random Effects Models, Sankhyā 46, Series B, 188-200.

Tietjen, G. L. (1974), Exact and Approximate Tests for Unbalanced Random Effects Designs, Biometrics 30, 573-581.

Tietjen, G. L. and R. H. Moore (1968), On Testing Significance of Components of Variance in the Unbalanced Nested Analysis of Variance, Biometrics 24, 423-429.

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